Orthogonal Surfaces

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Abstract. Orthogonal surfaces are nice mathematical objects which have interesting connections to various fields, e.g., integer programming, monomial ideals and order dimension. While orthogonal surfaces in one or two dimensions are rather trivial already the three dimensional case has a rich structure with connections to Schnyder woods, planar graphs and 3-polytopes.

Our objective is to detect more of the structure of orthogonal surfaces in four and higher dimensions. In particular we are driven by the question which non-generic orthogonal surfaces have a polytopal structure.

We study characteristic points and the cp-orders of orthogonal surfaces, i.e., the dominance orders on the characteristic points. In the generic case these orders are (almost) face lattices of polytopes. Examples show that in general cp-orders can lack key properties of face lattices. We investigate extra requirements which may help to have cp-orders which are face lattices.

Finally, we turn the focus and ask for the realizability of polytopes on orthogonal surfaces. There are criteria which prevent large classes of simplicial polytopes from being realizable. On the other hand we identify some families of polytopes which can be realized on orthogonal surfaces.

Mathematics Subject Classifications (2000). 05C62, 06A07, 52B05, 68R10.

1 Introduction

Subsection 1.1 is a short survey of previous work and important problems in the field of orthogonal surfaces. Subsection 1.2 is a collection of basic definitions and notation.

Section 2 is a review of 3-dimensional surfaces. We briefly look at the generic case and then move on to non-generic surfaces. In this still well visualizable case the distinction between generated and characteristic points becomes obvious and degeneracies can break the otherwise nice properties. Rigidity is the extra condition which helps.

Section 3 relates orthogonal surfaces and order theory. We discuss Schnyder's characterization of planar graphs and the Brightwell-Trotter Theorem in their relation with orthogonal surfaces and explain how dimension theory can help to prove that certain polytopes are not representable on orthogonal surfaces.

While the first three section mainly collect and review what was already known the final two sections contain new material.

With Section 4 we move on to higher dimensions. Issues of degeneracy and the relation between generated and characteristic points are analyzed with care. Characteristic points are also compared to the algebraically interesting syzygy-points of a surface. The concept of rigidity of an orthogonal surface is generalized to higher dimensions. Two concrete examples show that even in the rigid case cp-orders of 4-dimensional orthogonal surfaces may lack simple properties required for face-lattices of polytopes.

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Section 5 deals with realizability of polytopes on orthogonal surfaces. We present a new realizability criterion for simplicial polytopes. Exhaustive computations show that this criterion works for 2344 out of the 2957 simplicial balls on 9 vertices which are obtained by deleting a facet of a non-realizable polytope. In the final subsection we identify some families of realizable polytopes.

1.1 Previous work, motivation

Orthogonal surfaces have been studied by Scarf [16] in the context of test sets for integer programs. Initiated by work of Bayer, Peeva and Sturmfels [3] they later became of interest in commutative algebra. A recent monograph of Miller and Sturmfels [15] presents the state of the art in this area. Miller [14] was the first to observe the connections between orthogonal surfaces, Schnyder woods and the Brigthwell-Trotter Theorem about the order dimension of polytopes. We will outline these connections in Sections 2 and 3 where we also review other applications of order theoretic results to orthogonal surfaces.

Before stating the Theorem of Scarf which can be regarded the most fundamental result in the field we briefly set the stage with the most important terms.

Our starting point is a (finite) antichain V in the dominance order on $\mathbb{R}^d$. The *orthogonal surface* S_V generated by V is the topological boundary of the filter $\langle V\rangle = \big\{x \in \mathbb{R}^d :$ there is a $v \in V$ with $v_i \leq x_i$ for all $i\big\}$. *orthogonal surface*

An orthogonal surface S_V in $\mathbb{R}^d$ is *suspended* if V contains d extremal vertices. An orthogonal surface S_V is *generic* if no two points in V have a coordinate in common. *suspended* *generic*

The *Scarf complex* Δ_V of a generic orthogonal surface S_V generated by V consists of all the subsets U of V with the property that $\bigvee_{u\in U} u \in S_V$. It is a good exercise to show that Δ_V is a simplicial complex. *Scarf complex*

Theorem 1 (Scarf '73). *The Scarf complex Δ_V of a generic suspended orthogonal surface S_V in $\mathbb{R}^d$ is isomorphic to the face complex of a simplicial d-polytope with one facet removed.*

A proof of the theorem is given in [3]. Figure 1 shows an example. The dimension 3 case of Scarf's theorem was independently discovered by Schnyder [17].

An interesting problem inspired by Scarf's theorem is the realization question, asking for a characterization of those simplicial d-polytopes which have a corresponding orthogonal surface. We come back to this question in Sections 3 and 5.

The subject becomes much more complicated if we consider non-generic surfaces. In this case, it is not even clear how to define an appropriate complex on the vertex set V. To overcome this difficulty, we introduce an alternative interpretation of the Scarf-complex. We observe that every element $U \in \Delta_V$ corresponds to a *characteristic point* $p_U = \bigvee(U) \in S_V$. A more general definition of characteristic points is given in Section 2. For now, it is sufficient to think of them as the corners of the staircase. *characteristic point*

The *cp-order* of an orthogonal surface is the set of characteristic points equipped with the dominance order together with artificial 0 and 1 elements. With this terminology, we can rephrase Scarf's theorem as follows: *cp-order*

Theorem 2 (Scarf '73). *The cp-order of a generic suspended orthogonal surface is isomorphic to the face-lattice of some simplicial d-polytope with one facet removed.*

If the cp-order is a lattice, we will call it a *cp-lattice*. Scarf's Theorem implies that this is always the case if S_V is generic. *cp-lattice*

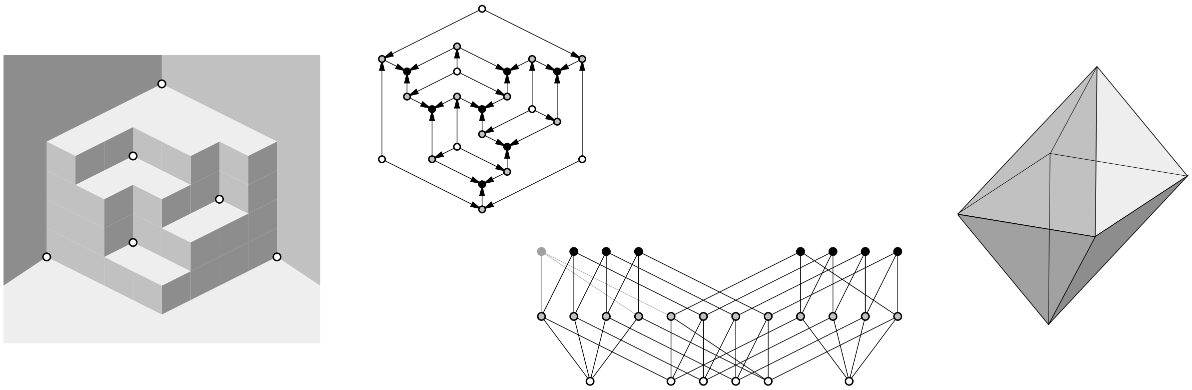

Figure 1: An orthogonal surface, two diagrams of its complex and the corresponding polytope.

One of our main goals is to determine conditions that are less restrictive than genericity but still guarantee that the cp-order has strong properties. In general, cp-orders of non-generic surfaces are no lattices, not graded and do not satisfy the diamond-property. We will discuss examples in Sections 2 and 4. To deal with these problems for 3-dimensional orthogonal surfaces, Miller introduced the notion of *rigidity*, [14]. We will define and discuss this property in Section 2 and its generalization to higher dimensions in Section 4. *rigidity*

The following theorem comprises a generalization of Scarf's theorem and the solution for the realization problem for the 3-dimensional case:

Theorem 3. *The cp-orders of rigid suspended orthogonal surfaces in* $\mathbb{R}^3$ *correspond to the face-lattices of* 3*-polytopes with one facet removed.*

In particular, rigidity implies that the cp-order is graded and a lattice. This result can be regarded as a strengthening of the Brightwell-Trotter Theorem [4] about the order dimension of face lattices of 3-polytopes (Theorem 6). Proofs can be found in [5] and [10]. These proofs actually show more, namely a bijection with Schnyder woods. We review some aspects of the theory in Section 2.

For dimensions $d > 3$ and non-generic surfaces, it is already challenging to come up with appropriate combinatorial definitions for characteristic points and properties of the cp-order. We present some results in Section 4. The dream which originated this research was to obtain some high-dimensional generalization of Theorem 3. The dream did not become true but we have shaped some basic blocks of theory which should have future.

1.2 Basic notation and definitions

We consider $\mathbb{R}^d$ equipped with the *dominance order*, this is the partial order on the points defined by the product of the orders of components, i.e. for $v, w \in \mathbb{R}^d$: *dominance order*

$$(v_1, v_2, \ldots, v_d) \leq (w_1, \ldots, w_d) \quad \Longleftrightarrow \quad v_i \leq w_i \text{ for all } i \in \{1, \ldots, d\}.$$

We say that v *strictly dominates* w if $v_i \gneqq w_i$ for all $i = 1, \ldots, d$ and denote this relation by $v \rhd w$. *strictly dominates*

A point v *almost strictly dominates* another point w, if $v_i = w_i$ for exactly one coordinate i and $v_j \gneqq w_j$ for all $j \neq i$, we denote this with $v \rhd_i w$. *almost strictly dominates*

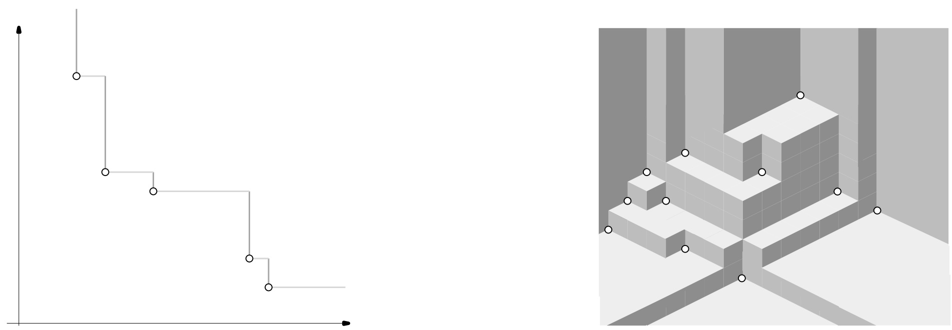

Figure 2: Orthogonal surfaces in two and three dimensions.

The *join* $v \vee w$ of points v and w is defined as the componentwise maximum of v and w and the *meet* $v \wedge w$ as the componentwise minimum of v and w. *join* *meet*

The *cone* $C(v)$ of $v \in \mathbb{R}^d$ is the set of all points greater than v in the dominance order, formally $C(v) = \{x \in \mathbb{R}^d \mid x \geq v\}$. *cone*

An *antichain* $V \subset \mathbb{R}^d$ is a set of pairwise incomparable points. This means for any $v, w \in V$, there are two coordinates $i, j \in \{1, \ldots, d\}$ such that $v_i < w_i$ and $v_j > w_j$. Equivalently, no point of V is contained in the cone of any other. *antichain*

The *filter* $\langle V \rangle$ generated by V is the union of all cones $C(v)$ for $v \in V$. *filter*

The *orthogonal surface* S_V generated by V is the boundary of $\langle V \rangle$. The generating set V is an antichain exactly if all elements of V appear as minima on S_V. We will generally assume that this is the case. *orthogonal surface*

A point $p \in \mathbb{R}^d$ belongs to S_V if and only if there is a vertex $v \in V$ such that $v \leq p$ and there is *no* $w \in V$ such that $p \rhd w$. In other words, S_V consists of points that share some coordinate with every vertex $v \in V$ they dominate.

With a point $p \in S_V$, we associate a *down-set* $D_p = \{v \in V : v \leq p\}$. For a point p and $v \in D_p$ define $T_p(v) = \{i \in \{1, \ldots, d\} : p_i = v_i\}$, this is the set of *tight coordinates* of p with respect to v. *down-set* *tight coordinates*

An orthogonal surface S_V in $\mathbb{R}^d$ is *suspended* if V contains a suspension vertex for each i, i.e., a vertex with coordinates $(0, \ldots, 0, M_i, 0, \ldots, 0)$, for each i and the coordinates of each non-suspension vertex $v \in V$ satisfy $0 \leq v_i < M_i$. *suspended*

An orthogonal surface S_V is *generic* if no two points in V have the same ith coordinate, for any i. If S_V is suspended then the condition has to be relaxed for the suspension vertices which obviously share coordinates of value zero. *generic*

The *Scarf complex* Δ_V of a generic orthogonal surface S_V generated by V consists of all the subsets U of V with the property that $\bigvee_{u \in U} u \in S_V$. *Scarf complex*

Figure 2 shows orthogonal surfaces in two and three dimensions. The picture on the right is obtained as orthogonal projection onto the plane $x_1 + x_2 + x_3 = 0$.

2 The 3-dimensional case

In this section we discuss 3-dimensional orthogonal surfaces. Before turning to the general case it seems appropriate to review the main correspondence in the generic case. We start with a correspondence between characteristic points of the surface and elements of the Scarf-complex:

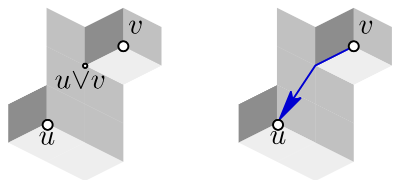

Figure 3: Drawing and orienting edges on S_V.

- Rank 0 elements of the complex (vertices) correspond to the minima of the surface, i.e, to elements of V.
- Rank 1 elements of the complex (edges) correspond to those elements of the surface which can be written as join $u \vee v$ for a pair u, v of vertices.
- Rank 2 elements of the complex (faces) correspond to the maxima of the surface, alternatively, these elements are joins of triples of vertices.

2.1 3-Dimensional and generic

Let S_V be a generic suspended orthogonal surface in $\mathbb{R}^3$, i.e., no two non-suspension points in V have a coordinate in common. We identify the coordinates 1,2,3 with the colors red, green and blue, in this order. In addition, we assume a cyclic structure on the coordinates such that $i+1$ and $i-1$ is always defined.

It is valuable to have a notation for some special features of the surface. For a vertex $v \in V$ and a color i define the *flat*[*] $F_i(v)$ as the set of points on S_V which dominate v and share coordinate i with v. The intersection $F_{i-1}(v) \cap F_{i+1}(v)$ of two flats of v is the *orthogonal arc* of v in color i.

flat

orthogonal arc

Draw every rank 1 element $\{u, v\}$ of the complex Δ_V as combination of two straight line segments, one connecting u to $u \vee v$, the other connecting v to $u \vee v$. This yields a drawing of a graph on S_V. Before discussing properties of the graph we impose additional structure on these edges.

For two vertices u and v (at most one of them a suspension vertex) genericity implies that the join $u \vee v$ has one coordinate from one of the them and two of the coordinates from the other vertex. In particular this is true if $u \vee v \in S_V$, i.e, if u, v is an edge in Δ_V. If $u \vee v \in S_V$ and $u \vee v$ has two coordinates from v then we orient the edge as $v \to u$ and color it with color of the coordinate which comes from u. In Figure 3 $u \vee v = (v_1, v_2, u_3)$ and the edge is oriented $v \to u$ and colored with color 3. The drawn edge consists of the orthogonal arc of v in color 3 which leads from v to $u \vee v$ and a segment between $u \vee v$ and u which traverses the flat $F_3(u)$.

From the geometry of flats (in the generic case a flat only contains a single vertex $v \in V$) and the way edges are drawn we can conclude the following:

- There are no crossing edges, i.e, the graph is planar.
- Every maximum of the surface dominates exactly three vertices, i.e., the graph is a triangulation.
- The orientation and coloring of the edges has the following properties:

  [Rule of Vertices] Every non-suspension vertex v has one outgoing edge in each color. The out-edges e_1, e_2, e_3 with colors 1, 2, 3 leave v in clockwise order. Each edge

[*]The definition given here is only valid in the generic case!

entering v with color i enters in the clockwise section from e_{i+1} to e_{i-1}. Suspension vertices only have incoming edges of one color.

The 'rule of vertices' defines a *Schnyder wood* of a planar triangulation. *Schnyder wood*

This explains how to obtain a Schnyder wood on a planar triangulation from a generic suspended orthogonal surface in $\mathbb{R}^3$. For the converse consider a triangulated planar graph. Selecting an outer triangle yields an essentially unique plane embedding. Specify a Schnyder wood of the plane triangulation – it was shown by Schnyder [17] that these structures exist, actually, a triangulation can have many different Schnyder woods, see [7].

The set of all edges of color i forms a directed tree spanning all interior vertices of the triangulation, this tree is rooted at one of the three outer vertices which will be called the *suspension vertex* of color i. The three trees define three colored paths $P_1(v)$, $P_2(v)$ and $P_2(v)$ from an interior vertex v to the three outer vertices. From the 'rule of vertices' it can be deduced that these paths are interiorly disjoint. Hence, they partition the interior of the outer triangle of the graph into three regions $R_1(v)$, $R_2(v)$ and $R_3(v)$. The *region vector* of a vertex v is the vector (v_1, v_2, v_3) defined by *suspension vertex* *region vector*

$$v_i = \text{The number of faces contained in region } R_i(v).$$

The set of region vectors of vertices of the graph yields a finite antichain $V \subset \mathbb{R}^3$ such that the orthogonal surface S_V has a complex Δ_V which is isomorphic to the original plane triangulation. Moreover, the orientation and coloring of edges on the surface S_V induced the Schnyder wood used for the construction of the surface.

Some of the details of the proof can be found in the original papers of Schnyder [17, 18], the notion of an orthogonal surface, however, was not known to Schnyder. Proofs given in the publications [14, 5] and in the book [6] extend these ideas to the more general case of 3-connected planar graphs.

By Steinitz's Theorem planar triangulations are essentially the same as simplicial 3-polytopes. Disregarding a facet of a 3-polytope corresponds to a choice of the outer face for the corresponding planar graph. Therefore, the following proposition is a colored strengthening of special cases of Theorem 1 and Theorem 3:

Proposition 1. *The edge colored complex of a generic suspended orthogonal surface S_V in $\mathbb{R}^3$ is the Schnyder wood of a plane triangulation. Moreover, every Schnyder wood of a plane triangulation has a corresponding orthogonal surface.*

In the above sketch we have been using Schnyder woods. In [18] Schnyder introduced *angle labelings* of plane triangulations and proved that they are in bijection with Schnyder woods. The two properties of angle labelings are *angle labelings*

[Rule of Vertices] The labels of the angles at each vertex form, in clockwise order, a non-empty interval of 1's, 2's and 3's.

[Rule of Faces] The labels in each face are 1, 2, 3 in clockwise order.

From an orthogonal surface supporting a Schnyder wood the corresponding angle labeling is directly visible: The angle between consecutive edges e and e' at vertex v is colored i if both edges leave v on the flat $F_i(v)$. Figure 4 shows a graph on a surface with the induced edge and angle colorings.

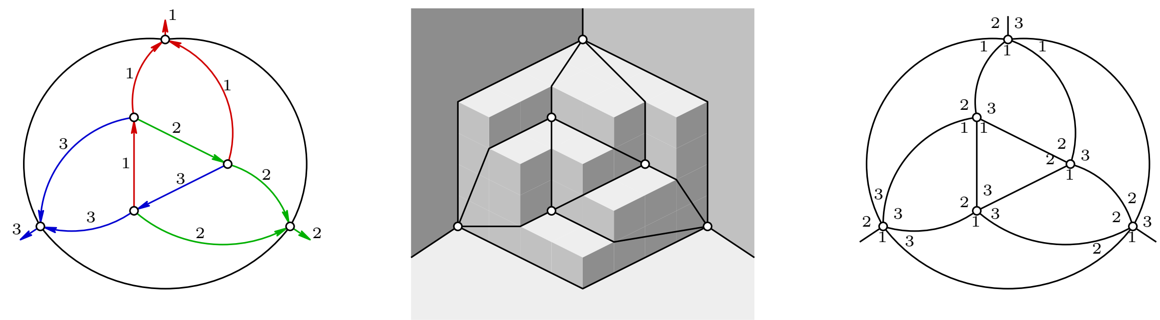

Figure 4: The graph of an orthogonal surface and corresponding Schnyder's colorings.

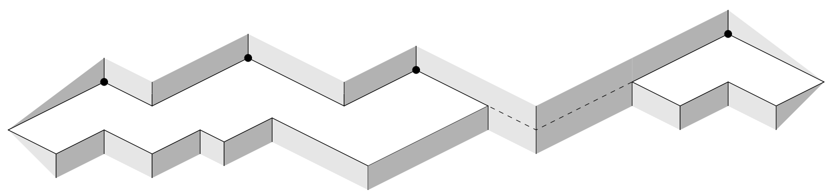

Figure 5: Two flats with the same defining coordinate, one with three minima and one with a single minimum.

2.2 3-Dimensional and non-generic

Given a non-generic antichain V in $\mathbb{R}^3$ it would be nice to have a complex Δ_V such that the elements of the complex are in bijection with the characteristic points of the surface S_V, just as in the generic case. Attempts to define such a Δ_V face some problems.

First of all, we have to rework and generalize our notion of a flat. Instead of attaching a flat strictly to one minimum, we now think of flats as connected $(d-1)$-dimensional components of the intersection of S_V with some hyperplane. In the non-generic case such a component/flat can contain several minima, all sharing the coordinate, which defines the flat, see Figure 5.

For every $v \in V$ and every coordinate i, the *almost strict upset* $U_i(v) = \{p \in S_V : p \rhd_i v\}$ belongs to the same i-flat as v. *almost strict upset*

If $U_i(v) \cap U_i(w) \neq \emptyset$, then v and w belong to the same i-flat. More general, we define a relation $\sim_i$ on V by $v \sim_i w \Leftrightarrow U_i(v) \cap U_i(w) \neq \emptyset$. The transitive closure $\sim_i^c$ of $\sim_i$ is an equivalence relation. The equivalence classes are exactly those sets of minima sharing a common i-flat.

Definition 1. *Let $v \in V$. The i-flat $F_i(v)$ is the topological closure of the set* *i-flat*

$$\bigcup_{w \sim_i^c v} U_i(w)$$

The equivalence class of minima on an i-flat F_i is $V_{F_i} = F_i \cap V$. Furthermore, we define the *upper part of the flat* F_i as $F_i^u = \bigcup_{v \in V_{F_i}} U_i(v) = \{p \in S_V : p \rhd_i v \text{ for some } v \in V_{F_i}\}$ *upper part of the flat*

2.2.1 Degeneracies

There may be characteristic points which can be obtained as join of distinct pairs of vertices, e.g., $p = u \vee v = v \vee w = w \vee u$ for distinct vertices u, v, w. Figure 6 shows an example. We

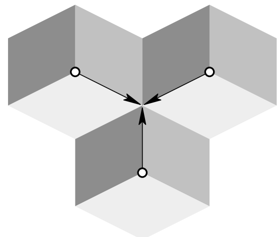

Figure 6: A degenerate situation on an orthogonal surface.

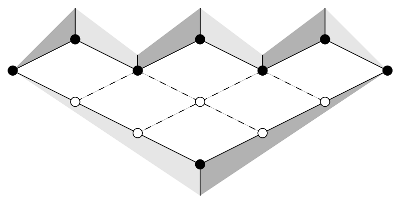

Figure 7: Characteristic and non-characteristic joins

want to have the property that every orthogonal arc is part of an edge, i.e., connects a vertex with a characteristic point of rank 1. Therfore, we usually assume that surfaces in $\mathbb{R}^3$ have no such substructure, if we want to emphasize this property we say the surface is *non-degenerate*. *non-degenerate*

2.2.2 Generated versus characteristic points

In the generic case every joint $u \vee v$ on the surface S_V is a characteristic point and corresponds to a rank 1 element of Δ_V. This is not true in general. An example is shown in Figure 7 where characteristic points are black but there are additional (white) generated points.

This shows the need of a new definition for characteristic points. In dimension three we could stick to the definition that characteristic points of rank 1 are endpoints of orthogonal arcs while all other characteristic points are minima or maxima. More satisfactory and more appropriate for generalizations to higher dimensions is the following:

Definition 2. *A* characteristic point *is a point which is incident to flats of all colors.* *characteristic point*

Clearly, every minimum is a characteristic point. From the definition it is immediate that every characteristic point is a generated point, i.e. can be expressed as the join of some minima. Figure 8 shows the possible types of points on a surface. Characteristic points are those of types **a**, **d** and **e**. Type **e** is the forbidden degenerate substructure.

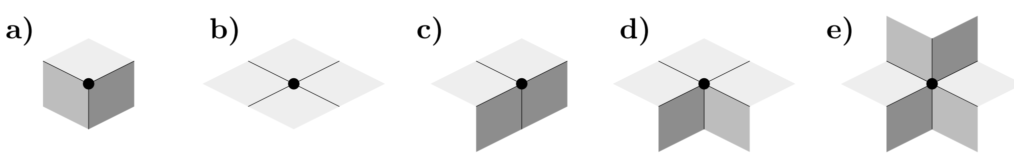

Figure 8: A classification of point types on orthogonal surfaces in $\mathbb{R}^3$.

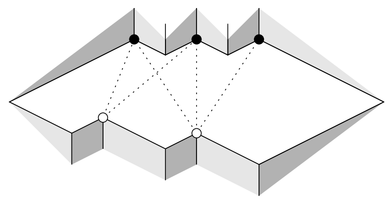

Figure 9: Characteristic points of rank 1 with two and three dominated vertices.

2.2.3 Rigidity

In the generic case the dominance order on characteristic points and the inclusion order of the sets of the complex Δ_V coincide. This is no longer true in the general case. Even characteristic point of rank 1 can dominate many vertices, see Figure 9.

In this case the graph defined by the surface is not unique. Uncoordinated choices for edges can even lead to crossing edges. Miller [14] calls a surface *rigid* if characteristic points $u \vee v$ only dominate u and v in V, i.e., there is no $w \in V \setminus \{u, v\}$ with $w \leq u \vee v$. *rigid*

Note that rigidity of a surface S_V in $\mathbb{R}^3$ implies that S_V is non-degenerate and that -as in the generic case- S_V defines a unique graph on the vertex set V: (u, v) is an edge if and only if $u \vee v$ is a characteristic point. Such an edge can be drawn as the combination of two straight line segments $u - u \vee v$ and $v - u \vee v$. At least one of them is an orthogonal arc of the surface and all orthogonal arcs emanating from vertices $v \in V$ are used. Again there is an obvious definition of orientations and colorings of edges.

Let G be a plane graph with *suspension vertices* a_1, a_2, a_3 on the outer face. Add a half-edge to each of the three suspension vertices. A *Schnyder wood* for G is an orientation and coloring of the edges with colors $1, 2, 3$ such that: *suspension vertices* *Schnyder wood*

(W1) Every edge e is oriented by one or two opposite directions. The directions of edges are labeled such that if e is bioriented, then the two directions have distinct labels.

(W2) The half-edge at a_i is directed outwards and labeled i.

(W3) Every vertex v has outdegree one in each label. The edges e_1, e_2, e_3 leaving v in labels 1,2,3 occur in clockwise order. Each edge entering v in label i enters v in the clockwise sector from e_{i+1} to e_{i-1}. [Rule of vertices]

(W4) There is no interior face whose boundary is a directed cycle in one label.

The orientation and coloring of edges induced by a suspended rigid orthogonal surface is a Schnyder wood for the induced plane graph.

It can be shown that a plane graph G with suspension vertices a_1, a_2, a_3 has a Schnyder wood exactly if the graph G_∞ which is obtained from G by adding a new vertex adjacent to the three suspension vertices is 3-connected.

Let G be a plane graph with a Schnyder wood. The edges of color i in the Schnyder wood induce a spanning tree rooted at a_i. These trees define paths and these paths, in turn, define three regions for every vertex. Therfore we can again consider the set V of region vectors of the vertices. This set V is an antichain in $\mathbb{R}^3$ and the surface S_V generated by V supports the graph G and the Schnyder wood which was used to define the regions. However, the surface S_V obtained by this construction needs not to be rigid.

Let S_V be constructed from the region vectors of a graph G with a Schnyder wood. The vertices, edges and bounded faces of G can be associated to the characteristic points of S_V.

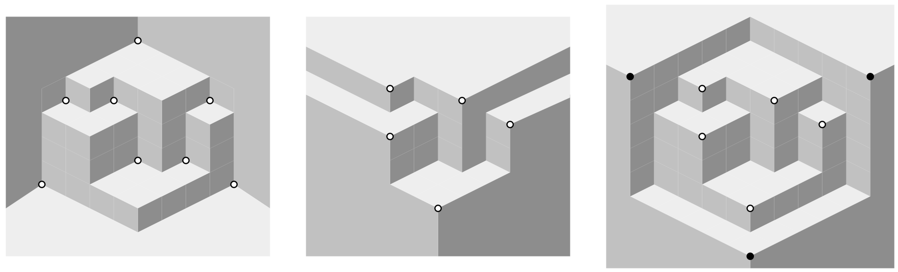

Figure 10: A rigid surface, the dual surface and the suspended dual surface.

Actually, it is even possible to associate with a Schnyder wood on a plane graph a *rigid* orthogonal surface which supports the Schnyder wood. Hence, there is an orthogonal surface which uniquely supports the given Schnyder wood. This has been conjectured by Miller [14] and was proven in [5] and [10], we come back to this in the next section.

With Steinitz's correspondence between 3-connected planar graphs and 3-polytopes we obtain Theorem 4. This theorem is a more precise restatement of Theorem 3.

Theorem 4. *The cp-orders of rigid suspended orthogonal surfaces in* $\mathbb{R}^3$ *coincide with the face-lattices of 3-polytopes with one facet removed or with one vertex of degree three and the incident faces removed.*

2.2.4 Duality

Let V be the generating antichain for a rigid orthogonal surface S_V in $\mathbb{R}^3$ and let W be the set of maxima of S_V. Consider the reflection at $\mathbf{0}$ and let $\bar{W}$ be the image of W under this map. The orthogonal surface $S_{\bar{W}}$ turns out to be -almost†- the same surface as S_V with the reversed direction of the dominance order. The surface $S_{\bar{W}}$ is again rigid and supports a unique Schnyder wood. Well not quite, the surface $S_{\bar{W}}$ can have more than three unbounded orthogonal arcs. This can be repaired by adding three suspension vertices to the dual which bundle the unbounded orthogonal arcs. Figure 10 shows an example. A more detailed account to the duality of Schnyder woods can be found in [7].

One interesting aspect of the duality is that superimposing a Schnyder wood and its dual Schnyder wood induces a decomposition of the surface into quadrangular patches. Each of these patches is completely contained in a flat, i.e., we can associate a color with each patch. This yields a joint angle coloring of the underlying planar graph and the dual.

3 Orthogonal surfaces and order dimension

Every order $P = (X, \leq)$ can be represented as intersection of linear extensions. That is there are linear orders $L_1, \ldots, L_k$ with the following properties:

- If $x \leq y$ in P, then $x \leq y$ in each L_i, i.e., the L_i are linear extensions of P.
- If $x||y$ in P, then there are indices i and j such that $x < y$ in L_i and $y < x$ in each L_j.

A set of linear extensions representing P in this sense is called a *realizer* of P. The smallest number of linear extensions in a realizer of P is the *dimension* $\dim(P)$ of P. *realizer* *dimension*

†The difference between S_V and $S_{\bar{W}}$ is in the unbounded flats.

Let $L_1, \dots, L_k$ be a realizer of P. With every $x \in X$ associate a vector $(x_1, \dots, x_k) \in \mathbb{R}^k$, where x_i gives the position (coordinate) of x in L_i. This mapping of the elements of P to points of $\mathbb{R}^k$ embeds P into the dominance order of $\mathbb{R}^k$. Ore defined $\dim(P)$ as the minimum k such that P embeds into $\mathbb{R}^k$ in this way. To prove that the two definitions are equivalent it remains to show how to obtain a realizer from an order preserving embedding into $\mathbb{R}^k$. If the coordinates of all points in the embedding are pairwise different (general position), then the projections to the coordinate axes form a realizer. Otherwise let $Y \subset X$ be a set of points sharing a coordinate, e.g., $Y = \{x \in X : x_1 = a\}$. The order relation among points in Y is completely determined by the coordinates $2, .., k$. Let $L(Y)$ be any linear extension of the order induced by Y. Displace coordinate 1 of the elements in Y by tiny amounts such that their projection confines with $L(Y)$. Repeated perturbations of this type yield an embedding of P in $\mathbb{R}^k$ which is in general position.

The following proposition is evident:

Proposition 2. *Let X be a finite set of points on an orthogonal surface S_V in $\mathbb{R}^k$ and let $P = (X, \leq)$ be the dominance order on X, then* $\dim(P) \leq k$.

With a graph $G = (V, E)$ associate the incidence order P_G as the order on the set $V \cup E$ with relations $v \leq e$ iff v is one of the two vertices of e. Schnyder's celebrated characterization of planar graphs is the following:

Theorem 5 (Schnyder). *A graph G is planar iff* $\dim(P_G) \leq 3$.

It was known already to Babai and Duffus [2] that $\dim(P_G) \leq 3$ implies that G is planar. Schnyder contributed the other direction. A proof in our context can follow these steps: Add edges to G to produce a planar triangulation G^*. Using a Schnyder wood this triangulation can be embedded in a generic orthogonal surface S_V in $\mathbb{R}^3$. The dominance order on the characteristic points of S_V is isomorphic to the complex Δ_V which contains P_G as a suborder. From that the result follows with Proposition 2.

Actually, the above sketch shows that for planar triangulations the incidence order of vertices, edges and bounded faces has dimension at most 3. This was known to Schnyder (see [17]), it is the simplicial polytope case of the following generalization of Schnyder's Theorem.

Theorem 6 (Brightwell-Trotter). *Let P be the inclusion order of vertices, and faces of a 3-polytope, then* $\dim(P) = 4$. *The inclusion order of vertices, edges and all but one of the faces only has dimension 3.*

The first part is based on a lower bound for the dimension of face lattices of polytopes. If P is a d-polytope and $\mathcal{F}(P)$ is its face lattice, then $\dim(\mathcal{F}(P)) \geq d + 1$. Since all the critical pairs are between a maximal and a minimal element of $\mathcal{F}(P)$ the bound on the dimension already holds for the suborder induced by maximal and a minimal elements.

The second part follows from the existence of a rigid embedding of the corresponding graph on an orthogonal surface in $\mathbb{R}^3$.

3.0.5 Realizability and order theory

In the terminology developed in the meanwhile we can restate Scarf's theorem: The dominance order on characteristic points of a generic suspended surface in $\mathbb{R}^d$ is isomorphic to the face lattice of a simplicial d-polytope with one facet removed. This result motivates the following general question:

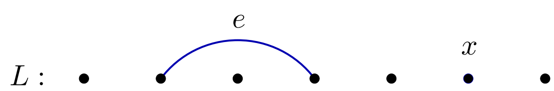

Figure 11: Vertex x is over edge e in L.

Problem 1 (Realizability Problem). *Which d-polytopes can be* realized *on an orthogonal surface in $\mathbb{R}^d$, i.e., which face lattices of d-polytopes, with one facet removed are cp-lattices?* *realized*

Order theory can provide some criteria for non-realizability.

The *dimension of the complete graph* K_n is the dimension of its incidence order. The asymptotic behavior of this parameter was first discussed by Spencer [19]. Trotter improved the lower bound. Their work implied that the dimension of the complete graph is closely related to the number of antichains in the subset lattice. This well studied problem is known as "Dedekind's Problem." Although no closed form answer is known, good asymptotic bounds are known, they suffice to show that *dimension of the complete graph*

$$\dim(K_n) \sim \log\log n + \big(1/2 + o(1)\big) \log\log\log n.$$

More recently Hoşten and Morris [12] could directly relate $\dim(K_n)$ to a specific class of antichains in the subset lattice. From this work we know the precise value of $\dim(K_n)$ for all $n \leq 10^{20}$, for example $\dim(K_{12}) = 4$ and $\dim(K_{13}) = 5$.

For all integers n there exist simplicial 4-polytopes with a complete graph as skeleton, i.e., the first two levels of their face lattice is the incidence order of a complete graph K_n. These polytopes are called neighborly (c.f. Ziegler [20]). From the dimension of complete graphs it follows that for $n \geq 13$ these 4-polytopes are not realizable on an orthogonal surface in $\mathbb{R}^4$.

A more general criterion was developed by Agnarsson, Felsner and Trotter [1]. They show that the number of edges of a graph with an incidence order of dimension 4 can be at most $\frac{3}{8}n^2 + o(n^2)$.

With increasing dimension d there is only a rather weak bound: From $\dim(K_r) > d$ it can be concluded that a graph of dimension d has at most $\frac{1}{2}(1 - \frac{1}{r})n^2$ edges. For $d = 5$ this gives a bound of $\frac{81}{164}n^2$ edges.

Orthogonal surfaces are completely determined by the position of their vertices. Therefore, the following notion for the *dimension of a graph*, seems to be more appropriate in our context. Let $G = (V, E)$ be a finite simple graph. A nonempty family $\mathcal{R}$ of linear orders on the vertex set V of graph G is called a *realizer* of G provided: *dimension of a graph* *realizer*

($*$) For every edge $e \in E$ and every vertex $x \in V \setminus e$, there is some $L \in \mathcal{R}$ so that $x > y$ in L for every $y \in e$.

The *dimension* of G, denoted $\dim(G)$, is then defined as the least positive integer t for which G has a realizer of cardinality t.

An intuitive formulation for condition ($*$) is as follows: For every vertex v and edge e with $v \notin e$ the vertex has to get over the edge in at least one of the orders of a realizer. All the above results about dimension of incidence orders of graphs carry over to this notion of dimension. Actually, the two concepts are almost identical:

- The dimension $\dim(G)$ of a graph equals the interval dimension of its incidence order P_G. In particular $\dim(G) \leq \dim(P_G) \leq \dim(G) + 1$ and $\dim(G) = \dim(P_G)$ if G has no vertices of degree 1 (see [9]).

Let L and L' be linear orders on a finite set X. We say that L' is the *reverse* of L and write $L' = L^{\mathrm{rev}}$ if $x < y$ in L if and only if $x > y$ in L for all $x, y \in X$. *reverse*

Definition 3. *For an integer $t \geq 2$, we say that the dimension of a graph is at most $[t-1 \updownarrow t]$ if it has a realizer of the form $\{L_1, L_2, \ldots, L_t\}$ with $L_t = L_{t-1}^{\mathrm{rev}}$. Similarly, the dimension is at most $[t-1 \updownarrow\updownarrow t]$ if it has a realizer of the form $\{L_1, L_2, \ldots, L_t\}$ with $L_t = L_{t-2}^{\mathrm{rev}}$ and $L_{t-1} = L_{t-3}^{\mathrm{rev}}$.*

One of the motivations for introducing this refined version of dimension was the following theorem proven in [9]. Again, Schnyder woods are the main ingredient to its proof.

Theorem 7. *A graph G is outerplanar iff it has dimension at most $[2 \updownarrow 3]$.*

There are some results concerning the extremal problem of maximizing the number of edges of a graph of bounded dimension. The first results from [9] only where asymptotic. Felsner [8] has obtained sharp bounds:

- A graph of dimension $[3 \updownarrow 4]$ has at most $\lfloor \frac{1}{4}n^2 + n - 2 \rfloor$ edges.
- A graph of dimension $[3 \updownarrow\updownarrow 4]$ has at most $\frac{1}{4}n^2 + 5n$ edges.

These bounds easily translate into bounds for the number of characteristic points of rank 1 on orthogonal surfaces in $\mathbb{R}^4$ which are generated by an antichain V with the additional property that certain pairs of coordinate-orders are reverse to each other.

4 Higher dimensional orthogonal surfaces

4.1 Degeneracies

On a three-dimensional orthogonal surface, there are three types of characteristic points: local minima, saddle points, and local maxima. This classification implies a geometric *rank-function* on the set of characteristic points. *rank-function*

In general, we aim for a combinatorial counterpart for this concept. In dimension three, points of different geometric rank are generated in different ways. In the generic case, the geometric rank of a point coincides with the number of minima below it.

Definition 4. *Let $g \in S_V$ be a generated point. A* generating set *for g is a set $G \subset D_g$ such that $\bigvee(G) = g$. A generating set G is* minimal *if $\bigvee(G \setminus \{v\}) < g$ for all $v \in G$.* *generating set* *minimal*

In the generic case, every characteristic point p has a unique (minimal) generating set, namely D_p, so we can simply define the rank as $r(p) = |D_p| - 1$. However, in general, there can be several minimal generating sets, and they can have different cardinalities, as illustrated in Figure 12. The following Lemma shows that such an undesirable situation can be recognized by a specific pattern in the sets of tight coordinates.

Lemma 1. *If there is a generated point $g \in S_V$ with two minimal generating sets of different size, then there are three minima $u, v, w \in D_g$ and two coordinates i and j such that if we restrict the characteristic vectors $t_g(.)$ of $T_g(.)$ to positions i and j, then we have the following pattern:*

$$\begin{array}{ccc} & i & j \\ t_g(x) & 1 & 1 \\ t_g(u) & 0 & 1 \\ t_g(v) & 1 & 0 \end{array}$$

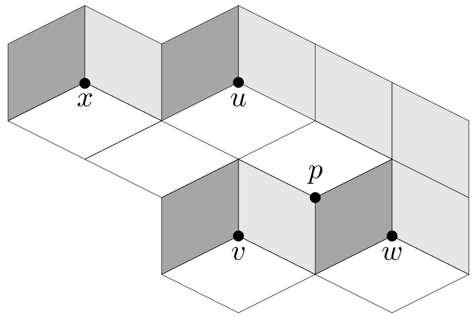

Figure 12: Point p has minimal generating sets $\{u, v, w\}$ and $\{x, w\}$.

In other words, $u_i < v_i = x_i = g_i$ *and* $v_j < u_j = x_j = g_j$

Proof. For every minimal generating set G and every $v \in G$, there is some coordinate i such that $w_i < v_i$ for all $w \in G \setminus \{v\}$, i.e., v is the only minimum contributing i. We call i the *private coordinate* of v in G.

Let G, G' be minimal generating sets for g such that $|G| > |G'|$. By the pigeon-hole principle, there is some $x \in G'$ covering private coordinates from at least two minima $u, v \in G$. □

The converse is almost true. If g is a generated point with the pattern from Lemma 1 and if V is suspended, then there is a characteristic point p with minimal generating sets of different sizes. Such a point p can be reached from g by increasing all coordinates except the two involved in the pattern. Since the surface is suspended, each increase is bounded. The point finally reached is contained in a flat of each color, hence, it is a characteristic point.

Definition 5. *The antichain V and the corresponding surface S_V are called* degenerate *if there is a characteristic point with the pattern shown in Lemma 1. Otherwise, V and S_V are* non-degenerate*.* *degenerate* *non-degenerate*

If V is non-degenerate, then all minimal generating sets of a characteristic point have the same cardinality. In this case, we define the *rank* of a characteristic point as the size of a minimal generating set minus one. Minima have rank 0 and maxima rank $d - 1$. *rank*

In the 3-dimensional case this definition does not only classify surfaces with a degenerate vertex as in Figure 6 as 'bad', but also some surfaces which support a proper planar graph, as in Figure 12.

A strong degeneracy is when two different i-flats intersect in their boundaries, as in Figure 6. From the 3-dimensional examples it seems plausible that degeneracies which are not strong could be removed by perturbing flats until different i-flats have different i-values, while the cp-order remains the same. The following example shows that this is not always possible.

We consider a weakly degenerate surface generated by four minima a, b, c, d. Different i-flats have different i-values, hence the characteristic point $p = (2, 2, 2, 2)$ is contained in exactly four flats. However, since pairs of minima below p share coordinates (and lie on common flats) in a cyclic structure, it is not possible to perturb these flats and remove the degeneracy.

$$a = (2, 2, 1, 1), \quad b = (1, 1, 2, 2), \quad c = (2, 1, 2, 1), \quad d = (1, 2, 1, 2)$$

We provide the details proving that a and c are contained in a common 1-flat. By the definition of a flat, it is sufficient to find a point $q \in S_V$ such that $q \rhd_1 a$ and $q \rhd_1 c$. The point $q = (2, 2+\epsilon, 2+\epsilon, 1+\epsilon)$ has the required properties.

4.2 Generated versus characteristic points

In the following, we assume that V is non-degenerate. In this case, we have a combinatorial criterion for characteristic points:

Proposition 1. *A generated point p is characteristic if and only if there are no minima $u, v \in D_p$ such that $T_p(u) \subset T_p(v)$.*

We will prove this proposition in five steps. Lemmas 2 and 1 yield a combinatorial criterion for the containment of a point p in a given flat F.

Lemmas 3 and 4 establish the connection between the subset-criterion and flat-containment. Finally, Lemma 5 shows that if V is non-degenerate and $p \in S_V$ is a characteristic point with $v \in D_p$, then $p_i = v_i \Longleftrightarrow p \in F_i(v)$.

Lemma 2. *Let $p \in S_V$. Let F be some i-flat of S_V. Then $p \in F$ if and only if there is a $v \in V_F$ and a $q \in S_V$ such that $q \rhd_i v$ and $v \leq p \leq q$.*

Proof. "$\Leftarrow$": If $p \rhd_i v$, then by definition $p \in U_i(v) \subset F_i(v)$. Otherwise, $p \neq q$ and there is a $q' \in [p, q] \subset [v, q] \subset S_V$, such that q' is arbitrarily close to p and $q' \rhd_i v$. Hence p is in the closure of $U_i(v)$ and thus, $p \in F_i(v)$.

"$\Rightarrow$": If p is in the upper part F^u of F, there is nothing to show , because then $p \rhd_i v$ for some $v \in V_F$ and $q = p$ is a good choice for q.

Now assume that $p \in F \setminus F^u$. Since p is in the closure of F^u, for every $\epsilon > 0$, there is a $q \in F^u$ such that $|p - q| < \epsilon$. In particular, $|p_j - q_j| < \epsilon$ for all coordinates j.

Claim 1. There is a $v \in V_F$ such that $p \geq v$.

Let $q \in F^u$ be close to p. By definition of F^u, there must be a $v \in V_F$ such that $q \rhd_i v$. Suppose $p \ngeq v$. Then there is a coordinate $j \neq i$ such that $p_j < v_j < q_j$. This is a contradiction to $|p_j - q_j| < \epsilon$ for $\epsilon < |p_j - v_j|$. In short, if q and p are close enough, then $q \rhd_i v$ implies $p \geq v$.

Claim 2. There is $q \in F^u$ with $q \geq p$.

We go for a contradiction and assume that there is none. It follows that all $x \in \mathbb{R}^d$ with $x \rhd_i v$ and $x \geq p$ are not on S_V. Hence, for every such x there is an *obstructor* $w \in V$ with $w \lhd x$. *obstructor*

Let γ be the vector with $\gamma_j = 1$ iff $v_j = p_j$ and $\gamma_j = 0$ iff $v_j < p_j$ and γ^i be obtained from γ by changing the value of coordinate i to 0. Consider the sequence $x_n = p + \frac{1}{n}\gamma^i$ converging to p. If each x_n has an obstructor then there has to be a simultaneous obstructor w for all elements of the sequence. From $w \lhd x_n$ for all n we obtain that if j is a coordinate with $\gamma^i_j = 1$, then $w_j \leq p_j = v_j$, and if $\gamma^i_j = 0$, then $w_j < p_j$. Hence, for all j either $w_j \leq v_j$ or $w_j < p_j$.

Let $q \in U_i(v)$ be close to p. Since q is not obstructed by w there is a coordinate j with $q_j \leq w_j$. Since $q \rhd_i v$ and because $w_j > v_j$ implies $p_j > w_j$ (previous paragraph), we have $v_j < q_j \leq w_j < p_j$. This is is impossible if $|p_j - q_j| < \epsilon$ and $\epsilon < |p_j - w_j|$. It follows that all points in F^u close enough to p are obstructed by w and, hence, $p \notin F$. This contradiction completes the proof. ◻

Let $p \in S_V$. If $v \in D_p$ is a minimum proving that $p \in F_i(v)$ in the sense of Lemma 2, i.e. there is a $q \in S_V$ such that $v \leq p \leq q$ and $q \rhd_i v$, then we call v an *i-witness* for p. *i-witness*

Obviously, if v is an i-witness for p, then $p \in F_i(v)$. The reverse is in general not true.

Corollary 1. *Given $p \in S_V$ and $v \in D_p$ with $p_i = v_i$, then v is an i-witness for p if and only if there is no minimum $w \in V$ such that $w_i < v_i = p_i$ and for all $j \neq i$ either $w_j \leq v_j$ or $w_j < p_j$.*

Proof. This follows from the proof of Claim 2 in the previous lemma. □

Lemma 3. *Let $p \in S_V$, $u, v \in D_p$, $T_p(u) \subset T_p(v)$, $i \in T_p(v) \setminus T_p(u)$. Then v is no i-witness for p.*

Proof. Assume otherwise, and let $q \rhd_i v$ and $q \geq p$. We show that this implies $q \rhd u$. This is a contradiction to $q \in S_V$. We have to check $q_j > u_j$ for all $j \in \{1, \dots, d\}$:

For all $j \notin T_p(u)$, we have $q_j \geq p_j > u_j$. This includes $j = i$. For all $j \in T_p(u)$, $j \neq i$, we have $q_j > v_j$, because $q \rhd_i v$, and $v_j = u_j = p_j$, because $T_p(u) \subset T_p(v)$. □

Lemma 4. *Let $p \in S_V$, $v \in D_p$, $v_i = p_i$, and assume v is not an i-witness for p. Then there is a minimum $u \in D_p$ such that $T_p(u) \subset T_p(v)$ and $i \in T_p(v) \setminus T_p(u)$.*

Proof. By Corollary 1, there is a $u \in V$ such that $u_i < v_i = p_i$ and for all $j \neq i$: $u_j \leq v_j$ or $u_j < p_j$. This implies that there is no coordinate k such that $v_k < u_k = p_k$. Therefore, $T_p(u) \subseteq T_p(v)$ and since $i \notin T_p(u)$ even $T_p(u) \subset T_p(v)$. □

Lemma 5. *Let $p \in S_V$, $v \in D_p$, $v_i = p_i$, but assume v is not an i-witness for p. Then either p is not a characteristic point or V is degenerate.*

Proof. Since v is not an i-witness for p, there is a $u \in D_p$ such that $u_i < v_i = p_i$ and $T_p(u) \subset T_p(v)$ by Lemma 4.

Assume p is characteristic. Then p is contained in some i-flat, hence there must be some i-witness w for p, and $T_p(u) \not\subseteq T_p(w)$ by Lemma 3. Therefore, there is a coordinate $j \neq i$, $j \in T_p(u) \setminus T_p(w)$. This results in the following pattern.

$$\begin{array}{rcllll} & & & i & j & \\ t_p(v) & = & (\dots & 1 & 1 & \dots) \\ t_p(u) & = & (\dots & 0 & 1 & \dots) \\ t_p(w) & = & (\dots & 1 & 0 & \dots) \end{array}$$

This shows that V is degenerate. □

Now we can complete the proof of Proposition 1:

Proof. If p is characteristic and $v \in D_p$, then by Lemma 5, v is an i-witness for p if and only if $p_i = v_i$. By Lemma 3, there can be no minima $u, v \in D_p$ such that $T_p(u) \subset T_p(v)$.

Conversely let p be generated, every coordinate of p is covered by some minimum. If there are no $u, v \in D_p$ such that $T_p(u) \subset T_p(v)$, then Lemma 4 implies that p is contained in every flat-type, hence, p is characteristic. □

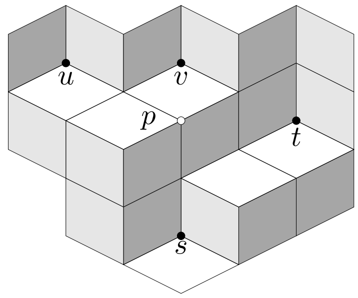

Figure 13: p has minimal generating sets $\{u, t\}$ and $\{v, s\}$

Proposition 1 implies that the minimal generating sets and the down-set D_p of a characteristic point p have a very special structure:

Corollary 2. *Let V be non-degenerate. A generated point $p \in S_V$ is characteristic of rank k if and only if there is a partition $P_1, \ldots, P_{k+1}$ of D_p such that $G \subset D_p$ is a minimal generating set for p if and only if $|G \cap P_i| = 1$ for all $i = 1, \ldots, k+1$.*

Two minima $u, v \in D_p$ belong to the same part P_i if and only if $T_p(u) = T_p(w)$.

Corollary 3. *Let V be non-degenerate and $p \in S_V$ be a characteristic point. Then every minimum $v \in D_p$ is contained in some minimal generating set of p.*

Observe that the corollary does not yield a criterion to distinguish characteristic points from generated points. There exist non-characteristic points such that every minimum below is contained in a minimal generating set, as in Figure 13.

4.3 Syzygy-points and characteristic points

Algebraists use orthogonal surfaces as a tool to obtain resolutions for monomial ideals. They are specially interested in the syzygy-points of a surface. In this subsection we discuss the relation between syzygy-points and characteristic points.

For every point $p \in S_V$, simplicial complex Δ_p on the set $\{1, \ldots, d\}$ is define by:

$$I \in \Delta_p \quad \Longleftrightarrow \quad p + \sum_{i \in I} \epsilon e_i \in S_V \quad \text{for some } \epsilon > 0.$$

A point p is a *syzygy-point* if Δ_p has non-trivial homology.

Lemma 6. *If p is a syzygy-point, then p is characteristic.*

Proof. Suppose p not characteristic. Then there is some $i \in \{1, \ldots, d\}$ such that p is not contained in any i-flat. This implies that there is some $\epsilon > 0$ such that $p + \epsilon e_i \in S_V$.

The claim is that in this case $I \in \Delta_p$ implies $I \cup \{i\} \in \Delta_p$. Let $I \in \Delta_p$ with $i \notin I$, let $p' = p + \sum_{j \in I} \epsilon\, e_j$. This point p' is on S_V by definition of Δ_p and p' is not contained in any i-flat if ϵ is small enough. This implies $p' + \epsilon\, e_i = p + \sum_{j \in I \cup \{i\}} \epsilon\, e_j \in S_V$. Therefore $I \cup \{i\} \in \Delta_p$.

It follows that every maximal simplex of Δ_p contains i, therefore, Δ_p is contractible. Hence, p is not a syzygy-point. $\square$

For $d = 3$, the converse is also true: every characteristic point is a syzygy-point. This can be verified by considering the types of points shown in Figure 8.

For $d \geq 4$, not all characteristic points are syzygy-points. We have an example of a characteristic point p that is no syzygy. Δ_p is simply a path on four vertices, hence contractible.

Example: A characteristic point which is not syzygy

The point p is generated by three minima with the following coordinates:

$$\begin{aligned} u &= (2,2,1,1) \\ v &= (2,1,2,1) \\ w &= (1,2,1,2) \end{aligned}$$

Let $p = u \vee v \vee w = (2,2,2,2)$. Clearly, $p \in S_V$, because p shares some coordinate with every minimum below.

We first check that p is indeed a characteristic point, using Lemma 2. For every i, we provide a minimum $m \leq p$ and a point $q \rhd_i m$ such that $m \leq p \leq q$, thereby proving $p \in F_i(m)$.

Each of the $q's$ in the following table shares one coordinate with each of the three minima u, v, w. This ensures that $q \in S_V$.

$$\begin{array}{lll} p \in F_1(u): & q = (2,3,2,2) \in S_V, & q \rhd_1 u \text{ and } u \leq p \leq q \\ p \in F_2(u): & q = (3,2,2,2) \in S_V, & q \rhd_2 u \text{ and } u \leq p \leq q \\ p \in F_3(v): & q = (3,2,2,2) \in S_V, & q \rhd_3 v \text{ and } v \leq p \leq q \\ p \in F_4(w): & q = (2,3,2,2) \in S_V, & q \rhd_4 w \text{ and } w \leq p \leq q \end{array}$$

Now we examine the simplicial complex Δ_p:

Let $\epsilon > 0$. For all $i = 1,2,3,4$, we have $p + \epsilon e_i \in S_V$, because p shares two coordinates with every minimum. This implies that Δ_p has the vertices 1, 2, 3, 4.

$$\begin{array}{ll} p + \epsilon e_1 + \epsilon e_2 \rhd u \Longrightarrow \{1,2\} \notin \Delta_p & p + \epsilon e_1 + \epsilon e_3 \rhd v \Longrightarrow \{1,3\} \notin \Delta_p \\ p + \epsilon e_1 + \epsilon e_4 \in S_V \Longrightarrow \{1,4\} \in \Delta_p & p + \epsilon e_2 + \epsilon e_3 \in S_V \Longrightarrow \{2,3\} \in \Delta_p \\ p + \epsilon e_2 + \epsilon e_4 \rhd w \Longrightarrow \{2,4\} \notin \Delta_p & p + \epsilon e_3 + \epsilon e_4 \in S_V \Longrightarrow \{3,4\} \in \Delta_p \end{array}$$

The complex Δ_p is a simple path on the vertices $1,2,3,4$, so Δ_p is contractible and hence has trivial homology. This proves that p is not a syzygy point.

4.4 Rigidity

Recall from the 3-dimensional case, that rigidity forces that a characteristic point of rank 1 can only dominate exactly two minima. An alternative formulation is that a characteristic point of rank 1 must not dominate another point of the same rank. This second condition can be generalized for characteristic points of arbitrary rank:

Definition 6. *An orthogonal surface S_V is called* rigid *if and only if the characteristic points of every rank are an antichain in the cp-order.* *rigid*

In other words, V is rigid if and only if the cp-order of V is *graded*. For face lattices of polytopes, this is a necessary condition. In particular, two faces of a polytope are comparable if and only if one is contained in the other, and this implies that they have different dimension. *graded*

For the three-dimensional case, rigidity is sufficient to ensure that the dominance order on characteristic points is indeed isomorphic to the face lattice of a 3-polytope (minus one facet), as we discussed in Section 2.

However, in dimension four, this is no longer true. There are examples showing that there remain rather substantial differences between cp-orders of rigid orthogonal surfaces and face lattices of polytopes in general:

- In a (face) lattice, any two elements have a unique join. There are rigid cp-orders that violate this condition and, hence, are no lattices (see Subsection 4.4.1).
- Even if the cp-order is a lattice it may have intervals of height 2 which are no quadrilaterals (see Subsection 4.4.2). This is impossible for face lattices of polytopes (face lattices of polytopes have the *diamond-property*).

Problem 2. *Identify further properties of cp-orders of (rigid) orthgonal surfaces.*

4.4.1 A rigid flat without the lattice-property

Figure 14 shows one flat F of an orthogonal surface in dimension 4. The surface is rigid. It is generated by four internal minima v, w, s, t with coordinates

$$v = (3,1,2,3), \quad w = (1,3,1,3), \quad s = (4,2,3,1), \quad t = (2,4,4,2)$$

together with the four suspensions X, Y, Z, T. The flat F is the flat $F_4(v) = F_4(w)$ In the figure the two minima v and w are marked black, characteristic points of rank 1 (edges) are marked white, points of rank 2 (2-faces) are marked blue and maxima are marked green.

Note that the boundary of F consists of a *lower staircase* containing v, w as well as some characteristic points of rank 1 and 2 and an *upper staircase* containing the edges (v, s) and (w, t) and all maxima of F. The maximum labeled M is generated by $\{v, w\}, s, t, Y$ (this is to be read as: M is minimally generated by the vertices s, t, Y together with either v or w).

Consider the interval $[v, M]$ in the dominance order. It is shown in the right part of the figure. All characteristic points in that interval are contained in F. Note that $[v, M]$ only contains two edges (v, s) and (v, w) and two 2-faces generated by $\{v, w\}, s, t$ and $\{v, w\}, s, Y$ respectively. Both edges are comparable to both 2-faces in the dominance order. This shows that the cp-order of the surface is not a lattice.

4.4.2 A rigid flat without the diamond property

Figure 15 shows a flat F of an orthogonal surface in dimension 4. The surface is rigid. It is generated by the four suspensions X, Y, Z, T together with six internal minima x, u, v, w, s, t:

$$x = (3,3,3,3),\ u = (1,4,4,3),\ v = (4,1,4,3),\ w = (4,4,1,3),\ s = (2,5,5,1),\ t = (5,5,2,1)$$

(For better visibility we have used a different set of coordinates in the figure. The combinatorial structure of the flat is not affected by this change.) The flat F contains four minima x, u, v, w, they share the last coordinate, so F is a 4-flat. Consider the characteristic point $p = (5,5,5,3)$ of rank 2. It is generated by the minima $\{x, u, v, w\}, s, t$ The interval $[v, p]$ contains three characteristic points of rank 1: $x \vee u$ and $x \vee v$ and $x \vee w$. They are all located on the lower staircase.

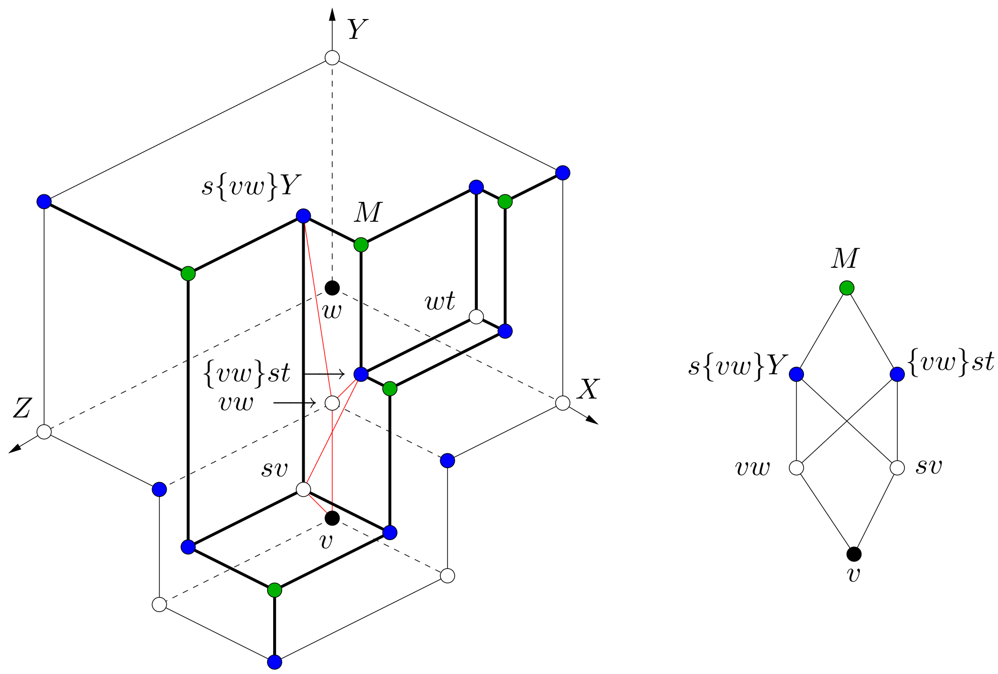

Figure 14: A rigid flat violating the Lattice-property

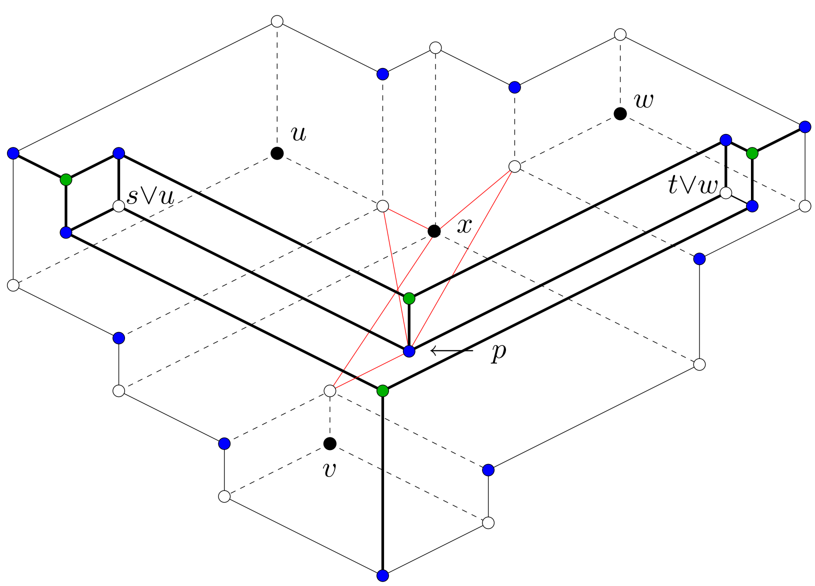

Figure 15: A rigid flat violating the Diamond-property

5 Realizability of polytopes

In the previous section we have investigated cp-orders. It became clear that even non-degenerate rigid surfaces can have cp-orders which are far from face lattices. In this section we turn the focus to polytopes and ask whether the face lattice of a given polytope P can be realized on an orthogonal surface.

In Section 3.0.5 we have seen realizability criteria which came from dimension theory of orders. The following subsection shows a criterion of different guise. After that we present some families of realizable polytopes.

5.1 Generic surfaces and realizability

The fact that every Scarf-complex is polytopal immediately raises the question whether every simplicial polytope is a Scarf-complex, or *realizable*. However, it is not difficult to find non-realizable simplicial polytopes. *realizable*

Every 3-polytope is a Scarf-complex, but already in dimension $d = 4$, there are large classes of non-realizable simplicial polytopes. One example, as mentioned in Section 3, are 4-polytopes with a skeleton-graph containing the complete graph K_n for $n \geq 13$. There are also smaller examples we will present later in this section.

A particularly well behaving class of polytopes are *stacked polytopes*. These are simplicial polytopes with the minimal number of faces. Every stacked polytope is realizable, we give a proof for this later in this section.

For arbitrary simplicial polytopes, there is no complete characterization of realizability. However, based on the combinatorial properties of cp-orders, we have a necessary criterion. It concerns the number of incidences between a $k-1$-face and a facet. Validating the criterion for a specific example only requires counting these incidences.

Proposition 2 (A Realization-Criterion). *Let $V \subset \mathbb{R}^d$ be a generic suspended antichain, and $p \in S_V$ be a characteristic point with $D_p = \{v_1, \ldots, v_k\}$. For every choice of coordinates*

$$i_1 \in T_p(v_1), \ldots, i_k \in T_p(v_k)$$

such that $p_{i_j} > 0$ for all $j = 1, \ldots, k$, there is a maximum $M \in S_V$ such that $M_{i_1} = p_{i_1}, \ldots, M_{i_k} = p_{i_k}$.

If S_V is suspended and p is an inner point, i.e. $p_i > 0$ for all i, then this implies that there are at least $|T_p(v_1)| \cdot |T_p(v_2)| \cdot \ldots \cdot |T_p(v_k)|$ maxima above p. In particular, the cp-lattice of S_V corresponds to a simplicial polytope where the face with vertices $\{v_1, \ldots, v_k\}$ is contained in at least $\prod_i |T_p(v_i)|$ facets.

Proof. The idea is to start at p and successively augment every coordinate $j \notin \{i_1, \ldots, i_k\}$ until we reach a maximum. The condition that no i_j is minimal ensures that we do not walk into one of the d unbounded flats.

We walk along the ray $p + \lambda e_j$ for $\lambda > 0$. We stop at the first point where $p + \lambda e_j \geq u$ for some minimum $u \notin D_p$. The point u is unique, because V is generic - there is no other minimum with this j-coordinate. Furthermore, $u_j \neq 0$ because $p_j < u_j$. Therfore, we can iterate with the point $p' = p \vee u$ with $D_{p'} = D_p \cup \{u\}$ and increase some other direction $j' \notin \{i_1, \ldots, i_k, j\}$. We can repeat the augmentation $d - k$ times. Finally we reach a point that dominates d minima and is minimal in no coordinate. This point shares exactly one

coordinate with every minimum below, so any further step in a positive direction would leave the surface. This characterizes a maximum M. The required properties of M are obvious: $M_j = p_j$ iff j is one of the selected coordinates $i_1, \ldots, i_k$. $\square$

We now take a closer look at a special case of this proposition and its applications. Let $d = 4$ and consider characteristic points generated by pairs of minima, i.e., edges.

In dimension 4, there are two possible forms for the join of two minima u, v. Either, $u \vee v$ inherits three coordinates from one of its generators and only one coordinate from the other, so w.l.o.g. $u \vee v = (u_1, v_2, v_3, v_4)$. The other case is that each generator contributes two coordinates, so w.l.o.g. $u \vee v = (u_1, u_2, v_3, v_4)$.

In the first case, (u, v) is the end-point of some orthogonal arc of v, and we call the edge (u, v) an *orthogonal edge*. In the second case, we call it a *symmetric edge*. For a symmetric edge (u_1, u_2, v_3, v_4), the criterion states that there are maxima $M_{1,3}, M_{1,4}, M_{2,3}, M_{2,4}$ such that $M_{i,j}$ inherits coordinate i from u and coordinate j from v. In particular, every symmetric edge has to be contained in at least four facets.

orthogonal edge

symmetric edge

A further important observation is that every inner vertex has exactly four outgoing orthogonal edges. This implies that in total, there are exactly $4n - 10$ orthogonal edges. All the other edges must be symmetric. In some cases, we can identify inner edges as symmetric, because we know that all edges connecting an inner point to a suspension point are orthogonal edges of the inner point.

These observations yield two useful methods to identify symmetric edges and thus prove the non-realizability of a complex:

- Any edge incident to a suspension vertex is orthogonal. Therefore, given two inner vertices v, w such that both are adjacent to *all four suspensions*, we know that the edge (v, w) is symmetric. ("Suspension-criterion")
- At least $|E| - (4n - 10)$ of the inner edges are symmetric. ("Counting-criterion")

We will consider two different but closely related aspects of the realization problem:

(A) Given a simplicial polytope, is there an cp-lattice realizing it?
This asks for the realization of a polytopal *sphere*?

(B) Given a simplicial polytope P with a designated facet F, is there an orthogonal surface realizing P such that F is the outer facet, i.e. the vertices of F are the suspensions?
This asks for the realization of a polytopal *ball*?

A polytope is non-realizable in the sense of (A) if and only if it is non-realizable for every choice of a facet F in the sense of (B). There are polytopes that are realizable in the sense of (A), but not for all possible choices of F. One example (already discussed in [3]) is the cyclic polytope $C_4(7)$. Here is a list of its facets:

[1, 2, 3, 4]	[1, 2, 3, 7]	[1, 2, 4, 5]	[1, 2, 5, 6]	[1, 2, 6, 7]	[1, 3, 4, 7]	[1, 4, 5, 7]
[1, 5, 6, 7]	[2, 3, 4, 5]	[2, 3, 5, 6]	[2, 3, 6, 7]	[3, 4, 5, 6]	[3, 4, 6, 7]	[4, 5, 6, 7]

The underlying graph is the complete graph K_7. This implies that no matter which facet we choose as the outer facet, every inner vertex must be adjacent to all four outer vertices,

i.e. the suspensions. The suspension criterion implies that all edges between inner vertices are symmetric.

The polytope $C_4(7)$ has two kinds of facets: If we choose one of the facets listed in the table below as the outer facet, then the remaining complex is not realizable. There is always an inner edge that is contained in only three facets.

Facet	$[1,2,3,4]$	$[1,2,3,7]$	$[1,2,6,7]$	$[1,5,6,7]$	$[2,3,4,5]$	$[3,4,5,6]$	$[4,5,6,7]$
Inner E.	$[5,7]$	$[4,6]$	$[3,5]$	$[2,4]$	$[1,6]$	$[2,7]$	$[1,3]$

If we choose any of the seven other facets, the remaining complex is realizable.

We have used a computer to generate a list of all orthogonal triangulations on 7, 8 and 9 vertices. This list has been compared to a list of all simplicial polytopes on these numbers of vertices[‡]. The comparison yields the following results:

- All simplicial polytopes on 7 and 8 vertices are realizable.
- On 9 vertices, there are 116 non-realizable simplicial polytopes in the sense of (A).

Every non-realizable polytopal sphere provides several non-realizable balls, because we can choose any facet as the outer facet. The 116 non-realizable polytopes on 9 vertices lead to 2957 non-realizable balls on 9 vertices. For these examples, we counted edge-facet incidences and compared to the two realization criteria. The results:

- 2141 of the 2957 non-realizable balls violate the "suspension-criterion" (816 do not).
- 2023 of the 2957 non-realizable balls violate the "counting-criterion" (934 do not).

Together, the two criteria work for 2344 of the balls. For the remaining 613 balls, counting the edge-facet incidences is not sufficient to prove that they are non-realizable.

For some of the difficult examples, new strategies to find symmetric edges might be sufficient to enable us to use the edge-facet criterion. For others, the edge-facet criterion is no help, because every edge belongs to at least 4 facets. For these cases, new arguments are needed.

5.2 Realizable polytopes

In this subsection we present classes of polytopes which can be shown to be realizable by an orthogonal surface. Recall that this means that the face lattice of the polytope is a cp-lattice. We have already mentioned classes of realizable simplicial polytopes, like all 4-polytopes with at most 8 vertices and *stacked polytopes.* These are polytopes that can be constructed from a simplex by a series of *stacking operations,* which means that a facet F is replaced by a vertex v and d new facets. In other words, a small pyramid with apex v is erected above F.

stacked polytopes

Proposition 3. *Every stacked polytope is realizable on a generic orthogonal surface.*

[‡]This list was compiled by Frank Lutz who also helped with the computations

Proof. A realization can be constructed inductively in the same way as the stacking, where a stacking operation corresponds to replacing a maximum M of the orthogonal surface S_V with a vertex v in the following way:

Assume M is generated by the d vertices $w_1, \ldots, w_d$, where w_i contributes the ith coordinate to M. We insert a minimum v with coordinates $v_i = w_{i_i} - \epsilon$, $i = 1, \ldots, d$. Obviously, $v \lhd M$.

For every $i \in \{1, \ldots, d\}$, there is a new maximum generated by v and all w_j's, $j \neq i$ to which v contributes the ith coordinate. These maxima correspond to the d new facets resulting from the stacking operation.

There are no characteristic points generated by v and any other vertices besides the w_i's. Assume there was such point $p > v$. Then p is strictly greater than v in some coordinate i. If ϵ is small enough, this implies $w_i \lhd p$, hence $p \notin S_V$.

$\square$

Proposition 4. *All d-polytopes on $d+2$ vertices are realizable.*

Proof (sketch). d-polytopes with $d+2$ vertices are completely classified, [11]. For every d, there are $\lfloor \frac{d}{2} \rfloor$ combinatorial types of simplicial d-polytopes with $d+2$ vertices. Non-simplicial d-polytopes on $d+2$ vertices are pyramids over some $d-1$-polytope on $d+1$ vertices.

A proof of the two following facts can be found in [13]:

- If a $d-1$-polytope P is a realizable on an orthogonal surface of dimension $d-1$, then the pyramid $pyr(P)$ is realizable in dimension d.
- There are $\lfloor \frac{d}{2} \rfloor$ combinatorially different orthogonal triangulations in dimension d on $d+2$ vertices.

The proposition follows by induction from these facts together with the realizability of all 3-polytopes. $\square$

Products of a polytope with an edge and more generally products with paths (*sequences*) preserve realizability. The construction is detailed in [13], here we only indicate the ideas:

Proposition 3. *Let P be a $d-1$-polytope. If P is a facet of some realizable d-polytope, then the prism over P, i.e. the product of P with an edge, is also a realizable d-polytope.*

Proof (sketch). Assume that P is realized as maximum M in an orthogonal surface of dimension d. The idea is to duplicate the local structure of M with slightly perturbed coordinates.

If v is a minimum contributing coordinate i to M, its double is $v' = v + 2\epsilon e_i - \epsilon\mathbf{1}$.

It is easy to check that the new vertices leave the old structure unchanged, i.e. they cannot obstruct any old faces. However, the join of the set of all new vertices is obstructed by M. Therefore, the counterpart-facet M' of M has no corresponding point on the surface. M' is the outer facet of the realization, i.e., the facet of the polytope which is missing in the cp-lattice of the surface. $\square$

Corollary 4. *The d-cube is realizable.*

The following more general construction produces a realization of a P-sequence, i.e., of the product of a realizable $d-1$-polytope P with a path: The realization of the product consists of translated copies of realizations of P:

Given a $d-1$-realization of P with minima $v_1, \ldots, v_n$, let v_j^i is the copy of vertex v_j in P^i. The coordinates of v_j^i are $(v_j - i\epsilon, i)$. For a small enough $\epsilon > 0$, the set $V_i = \{v_j^i : j = 1 \ldots n\}$ is an antichain. It is also easy to see that a vertex v_j^i is adjacent to only two vertices outside V_i, namely v_j^{i-1} and v_j^{i+1}.

It is an interesting question to identify further classes of realizable polytopes. Since the cyclic polytope $C(n,d)$ is not realizable for n sufficiently large we are particularly curious about the following:

Problem 3. *Are the dual polytopes of cyclic polytopes always realizable?*

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